

A Self-Contained Introduction to Khovanov Homology

Ambareesh Shyam Sundar

April 21, 2025

1 Introduction

In this paper, we aim to present a self-contained exposition of the key ideas of Khovanov homology. In particular, we assume no prior knowledge of category theory or knot theory (and only a basic background in homological algebra), and develop ideas such as the Jones polynomial from scratch. We then discuss how Khovanov homology arises as a natural extension, or “categorification”, of the Jones polynomial.

2 Background

Here we provide some background that is necessary to understand the exposition that is to follow. The reader is encouraged to skip this section and refer back to it as necessary.

2.1 Knots and Knot Invariants

In this section, we define our main object of study: knots.

Definition 2.1 (Knots and links). A *knot* is an embedding of S^1 into $\mathbb{R}^3$. We consider two knots equivalent if they are isotopic to each other – that is, one can be deformed into the other without passing one strand through another. More generally, we define a *link* as an embedding of some finite number of copies of S^1 into $\mathbb{R}^3$, with the aforementioned notion of equivalence.

Remark 2.2. Often, it is useful to consider a knot as an embedding of S^1 into S^3 instead of $\mathbb{R}^3$, due to the fact that S^3 is compact while $\mathbb{R}^3$ is not. In fact, many sources define knots in this way. For our purposes, we shall ignore this technicality due to the difficulty of visualizing embeddings into S^3 , and treat all knots as embeddings into $\mathbb{R}^3$.

It is a fundamental (and quite difficult) problem in knot theory to determine whether two knots are equivalent. In fact, it is even rather difficult to determine whether a given knot is equivalent to the unknot (whether there exists a polynomial-time algorithm to do so is still unknown). Therefore, to enable us to more easily differentiate knots, we develop *knot invariants*.

Definition 2.3 (Knot invariant). A *knot invariant* is a function f such that $f(K) = f(K')$ for any two equivalent knots K, K' .

It follows directly from the definition that if f is some knot invariant and $f(K_1) \neq f(K_2)$, then K_1 and K_2 are different knots (in the sense that there does not exist an isotopy between them).

Some examples of well known knot invariants are crossing number, tricolorability, and the Alexander polynomial. In this paper, we will focus on a knot invariant known as the *Jones polynomial* and its “categorification”, Khovanov homology.

The following is a classical result in knot theory. If two knot diagrams K and K' are equivalent, then there exists a sequence of *Reidemeister moves* that transforms K into K' . Therefore, to verify that a particular quantity is a knot invariant, it suffices to show that it is invariant under the Reidemeister moves.

Often, it will be useful to impose an *orientation* on a link. Informally, an orientation of a link is a “direction of travel” along each of its components. Oriented links are equivalent only if they are isotopic and have matching orientations.

Definition 2.4 (Positive and negative crossings). We can look at any oriented link diagram and assign to each crossing a *sign* (+ or −). Let D be an oriented link diagram and let c be a crossing in D . c is a *positive crossing* if, when the two strands are placed such that they are oriented upwards, the left strand goes over the right strand. Otherwise, c is a *negative crossing*.

2.2 Braid Groups

For a positive integer n , the symmetric group on n letters, denoted S_n , consists of all permutations of n elements. In particular, given any transposition $\sigma = (i\ j) \in S_n$, we have that $\sigma^2 = \text{id}$. What happens if we remove this property? Intuitively, we might expect that a transposition no longer behaves like a simple swap, but instead encodes some information about the *orientation* of the swap.

It turns out that this is exactly what defines the *braid group*.

Definition 2.5 (Braid group). Let n be a positive integer. The *braid group* on n strands, denoted B_n , is defined as

$$B_n \triangleq \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \rangle$$

The braid group has a nice pictorial representation. It is, exactly as one might guess, the set of braids on n strands of rope, considered up to isotopy where the ends of the strands are fixed. In particular, one can associate σ_i to the braid that swaps the i -th and $(i+1)$ -th strand, with the i -th strand going on top. When viewed from this perspective, multiplication of elements of B_n is equivalent to concatenating their corresponding braids.

3 The Jones Polynomial

To motivate the idea that Khovanov homology is a categorification of the Jones polynomial, we will step through the construction of this important knot invariant using the Kauffman bracket via the Temperley-Lieb algebra.

Remark 3.1. The Jones polynomial was originally defined in terms of representations of the quantum group $\mathfrak{sl}_2$. As the author is not so familiar with this field, we choose to take the alternative (and better known) approach given by Kauffman.

Keeping in mind that we seek to construct a knot (or link) invariant, we will take the following general approach. Given a diagram D of a link, we will first write it as the closure $\hat{b}$ of a braid $b \in B_n$. Then, we will resolve all the crossings in b via some consistent method that will produce an element of the Temperley-Lieb algebra $TL_n(A)$. Finally, we will determine a mapping from $TL_n(A)$ to $\mathbb{C}[A, A^{-1}]$.

3.1 The Temperley-Lieb Algebra

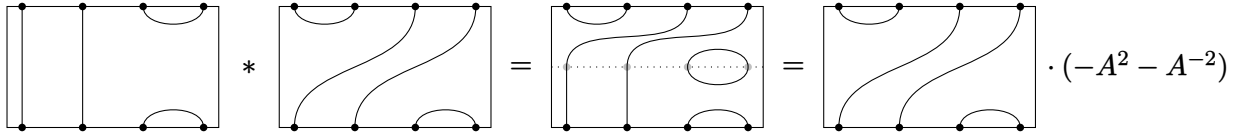
Definition 3.2 (Algebra over a commutative ring). Let R be a commutative ring. An *algebra over R* is an R -module M equipped with a binary operation $*$: $M \times M \rightarrow M$ such that $*$ is linear in each of its arguments.

For example, the complex numbers $\mathbb{C}$ with the standard multiplication form an algebra.

Definition 3.3 (Temperley-Lieb algebra). Let $R = \mathbb{C}[A, A^{-1}]$. The *Temperley-Lieb algebra* $TL_n(A)$ is the algebra $(M, *)$, where

- M is the set of R -linear combinations of *noncrossing vertical diagrams* from n points to n points (considered up to isotopy);
- For diagrams $a, b \in M$, $a * b$ is defined as the diagram obtained by stacking b on top of a ; and
- If a diagram a contains a circle and b is the same diagram with the circle removed, then $a = (-A^2 - A^{-2})b$.

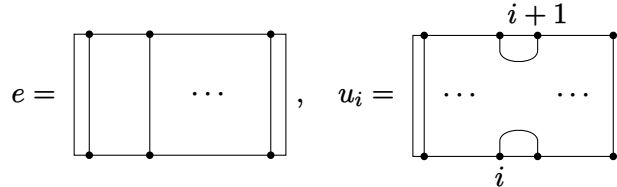
Example 3.4. We compute the product of two Temperley-Lieb diagrams in $TL_4(A)$:



Remark 3.5. There exist purely algebraic definitions of the Temperley-Lieb algebra. Our definition is equivalent, but the proof of this fact is a bit difficult (and not particularly important to the present discussion), so we omit it for the sake of brevity.

The way we have defined the Temperley-Lieb algebra makes some explicit choices about the conventions we will use. In particular, all diagrams will be drawn vertically, and stacked diagrams will be read bottom to top. Furthermore, we will define $d = -A^2 - A^{-2}$ to be the *loop constant* (for obvious reasons).

It is not too difficult to see that, for a given n , every diagram in $TL_n(A)$ can be written as a product of the generators $e, u_1, \dots, u_{n-1}$, where



That is, e is the identity, and u_i is the diagram with a cap-cup formation on strands i and $i + 1$.

Exercise 3.6. Show that, as a $\mathbb{C}[A, A^{-1}]$ -module, $TL_3(A)$ has rank 5. (*Hint: find all loop-free noncrossing diagrams, up to isotopy.*)

Remark 3.7. In general, the rank of $TL_n(A)$ as a $\mathbb{C}[A, A^{-1}]$ -module is equal to the n -th Catalan number; that is,

$$\text{rank } TL_n(A) = \frac{1}{n+1} \binom{2n}{n}.$$

For more details, consult [4].

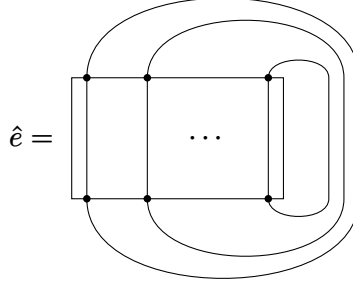
3.2 The Markov Trace

Now that we have defined the Temperley-Lieb algebra, we wish to construct a map from $TL_n(A)$ to $\mathbb{C}[A, A^{-1}]$. Since removing a loop from a diagram in $TL_n(A)$ is equivalent to multiplication by a factor of $d = -A^2 - A^{-2}$, one might think to turn all diagrams into unions of loops and then remove all of those loops to get a Laurent polynomial in q . In fact, there is a very natural way to turn a diagram into a union of loops.

Definition 3.8 (Closure). The *closure* $\hat{a}$ of a diagram $a \in TL_n(A)$ is the system of loops obtained by connecting corresponding strands in the top and bottom of the diagram.

Definition 3.9 (Markov trace). For a noncrossing diagram a on n points, let $\gamma(a)$ be the number of loops in $\hat{a}$. The *Markov trace* is the map $\text{Tr} : TL_n(A) \rightarrow \mathbb{C}[A, A^{-1}]$ that takes $a \in TL_n(A)$ to $d^{\gamma(a)}$ and is linear in its argument.

Example 3.10. The Markov trace of the identity $e \in TL_n(A)$ is d^n , since



has n loops.

Remark 3.11. If we take A to be an element of $\mathbb{C}$, then $\mathbb{C}[A, A^{-1}] = \mathbb{C}$ and $TL_n(A)$ becomes a vector space. We can use the trace to define an inner product on this vector space: if $a, b \in TL_n(A)$, set

$$\langle a, b \rangle = \text{Tr}(\bar{a}b)$$

where $\bar{a}$ represents a reflection of a across a vertical axis.

3.3 The Kauffman Bracket

We now have two of the three pieces required to construct our link invariant; we can take any diagram in $TL_n(A)$ to a Laurent polynomial in q with coefficients in $\mathbb{C}$. What is missing is a way to encode links as elements of $TL_n(A)$.

As a first step, we note that all Temperley-Lieb diagrams contain n distinct “strands”, whereas link diagrams contain just one “strand”. To remedy this, we can write any given link diagram as the *closure* of a braid in B_n .

Remark 3.12. Alexander’s theorem provides that any link diagram can be written as the closure of a braid. It is clear, however, that the map between links and braids is not bijective. If b and b' are two braids such that $\hat{b}$ and $\hat{b}'$ are the same knot, then b can be transformed into b' via a series of *Markov moves* (analogous to the Reidemeister moves for links). Specifically, the Markov moves are conjugation ($b \mapsto cbc^{-1}$) and stabilization ($b \mapsto b\sigma_n^{\pm 1}$).

Since the Temperley-Lieb algebra contains only noncrossing diagrams, we must deduce a way to eliminate crossings in link diagrams. We introduce the Kauffman bracket for this purpose.

Definition 3.13 (Kauffman bracket). Let D_n be the monoid consisting of diagrams (with crossings allowed) from n points to n points, where multiplication is given by stacking. Define an algebra homomorphism φ from the algebra $\mathbb{C}[A, A^{-1}]D_n$ to $TL_n(A)$ that satisfies the following relation:

$$\varphi(\times) = A \cdot \varphi(\succ) + A^{-1} \cdot \varphi(\swarrow)$$

(For brevity, we will refer to $\succ$ as the 0-resolution and $\swarrow$ as the 1-resolution of the crossing.) The *Kauffman bracket* $\langle \cdot \rangle$ is the map $b \mapsto \text{Tr}(\varphi(b))$.

3.4 Cleaning Up

At first glance, one might be tempted to believe that the Kauffman bracket is itself a link invariant. This is, unfortunately, not the case, as the next exercise will demonstrate.

Exercise 3.14 (Kauffman bracket of the unknot). Compute the Kauffman bracket of the following two diagrams whose closures are both isotopic to the unknot:

$$U_1 = \begin{array}{|c|} \hline \diagup \quad \diagdown \\ \hline \end{array}, \quad U_2 = \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \end{array}$$

and observe that the bracket polynomials are not equal.

To determine how to resolve this issue, we may observe how the Kauffman bracket changes under the Reidemeister moves. In particular, $\langle \cdot \rangle$ is invariant under Type II and Type III Reidemeister moves, but performing a Type I Reidemeister move multiplies the Kauffman bracket by a factor of $(-A^{\pm 3})$. Normalizing the bracket polynomial accordingly produces the long-awaited Jones polynomial.

Definition 3.15 (Jones polynomial). Let L be an oriented link, and suppose L is the closure of $b \in B_n$. The *unnormalized Jones polynomial* of L is defined as

$$\hat{J}(L) = (-A^{-3})^{w(L)} \langle L \rangle$$

and the *normalized Jones polynomial*, which we frequently refer to simply as the Jones polynomial, is

$$J(L) = \frac{\hat{J}(L)}{d}$$

where $w(L)$ is the *writhe* of L , defined as the number of positive crossings minus the number of negative crossings.

Remark 3.16. We divide by d in the definition of the Jones polynomial to set the Jones polynomial of the unknot to be equal to 1. It is conjectured that the Jones polynomial *detects* the unknot; that is, if $J(L) = 1$, the L is isotopic to the unknot. Surprisingly, this turns out to be **false** for links. For more information, see [2].

4 Khovanov Homology

Having discussed the basic construction of the Jones polynomial, we can now put on our category theory glasses. Now that we have developed the necessary machinery, producing the Khovanov homology turns out not to be so difficult.

4.1 A Notational Consideration

For reasons that will become apparent very soon, we define a new bracket operator on links, $\langle \cdot \rangle_*$, which is defined by $\langle L \rangle_* = A^{-c(L)} \langle L \rangle$ (where $c(L)$ is the number of crossings in the drawing of L). We then set $q = -A^{-2}$, so that the bracket relation becomes

$$\langle \diagdown \diagup \rangle_* = \langle \diagup \diagdown \rangle_* - q \langle \diagup \diagup \rangle_*$$

The loop constant is then $d = -A^2 - A^{-2} = q + q^{-1}$. Similarly, we can define the unnormalized Jones polynomial as

$$\hat{J}(L) = (-1)^{n-} q^{n_+ - 2n_-} \langle K \rangle_*$$

4.2 Motivation

To motivate the construction of Khovanov homology, we observe an important fact about the computation of the bracket polynomial: if L is a link diagram with n crossings, then $\langle L \rangle_*$ can be written as a sum of 2^n terms, each one corresponding to a specific resolution of the crossings in L . If we fix some particular ordering of the crossings $c_1, \dots, c_n$, there is a natural way to place these 2^n terms in a hypercube. This cube has vertex set $\{0, 1\}^n$, and a given $r = (r_1, \dots, r_n) \in \{0, 1\}^n$ corresponds to the Temperley-Lieb diagram in which crossing c_i has been given the r_i -resolution.

There is a natural notion of the “height” of a vertex in this cube: if $r = (r_1, \dots, r_n)$ has k entries equal to 1, then r has height $h(r) = k$. Let S_r be the link diagram L with crossings $c_1, \dots, c_n$ smoothed according to r . Note that each 1-smoothing contributes a factor of $-q$ to $\langle S_r \rangle_*$, and since S_r has no crossings, we may write

$$\langle S_r \rangle_* = (-q)^{h(r)} d^{r(S_r)}$$

We can stratify the cube by height, which gives us the nice property that, within a particular “height class” of smoothed diagrams, every term $\langle S_r \rangle_*$ has the same leading factor of $(-q)^{h(r)}$. Factoring out $(-1)^{h(r)}$ from each term gives an alternating sum over the height classes that yields the Jones polynomial.

4.3 Categorification

Those who are familiar with homological algebra will no doubt immediately recognize the way in which this definition of the Jones polynomial relates to chain complexes: this alternating sum over terms of consecutive “heights” is strongly reminiscent of the definition of the *Euler characteristic* of a complex! (Recall that given a chain complex $0 \rightarrow C_k \rightarrow C_{k-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow 0$, the Euler characteristic is defined as $\chi = \sum_n (-1)^n \text{rank } H_n$.) This gives us an idea on how to proceed: we want to construct a chain complex whose elements are somehow in correspondence with the vertices of the cube, and whose Euler characteristic is, in some sense, equal to the (unnormalized) Jones polynomial.

Observe that the hypercube of smoothings has an inherent 2-dimensionality to it: while the height of a given smoothing conveys information about its place in the chain complex, it does not say anything about the trace of the smoothing (and therefore how it will contribute to the Jones polynomial). It is natural, then, that we construct a chain complex on *bigraded* chain groups. The height of a given smoothing dictates its *homological degree*; on the other hand, the trace of the smoothing impacts its *quantum degree*. We now formalize these terms, taking inspiration from the notation of [1].

Definition 4.1 (Quantum degree). If $W = \bigoplus_m W_m$ is a graded vector space, then the *quantum degree* of W is defined as

$$\text{qdim } W \triangleq \sum_m q^m \dim W_m$$

Definition 4.2 (Quantum degree shift). If $W = \bigoplus_m W_m$ is a graded vector space, then we define the *quantum degree shift* operator $\cdot\{l\}$ by

$$W\{l\}_m \triangleq W_{m-l}$$

From the previous definition, it follows that

$$\begin{aligned}
\mathrm{qdim} W\{l\} &= \sum_m \mathrm{qdim} W\{l\}_m \\
&= \sum_m q^m W_{m-l} \\
&= q^l \sum_m q^{m-l} W_{m-l} \\
&= q^l \mathrm{qdim} W
\end{aligned}$$

Definition 4.3 (Homological degree shift). If $\mathcal{C}$ is a (co)chain complex $\cdots \rightarrow C_k \xrightarrow{\partial_k} C_{k+1} \rightarrow \cdots$, and $\bar{\mathcal{C}}$ is defined similarly with chain groups $\bar{C}_k$ and differentials $\bar{\partial}_k$, then we define the *homological degree shift* operator $\cdot[s]$ as follows: let $\mathcal{C} = \bar{\mathcal{C}}[s]$. Then $C_k = \bar{C}_{k-s}$, and $\partial_k = \bar{\partial}_{k-s}$.

Now that we have established that our chain groups will have a quantum grading, it makes sense to define an analogue of the standard Euler characteristic for our bigraded chain complex.

Definition 4.4 (Graded Euler characteristic). Let $\mathcal{C}$ be a chain complex with homology groups H_k . The *graded Euler characteristic* of $\mathcal{C}$ is defined as

$$\chi_q(\mathcal{C}) \triangleq \sum_m (-1)^m \mathrm{qdim} H_m$$

Of course, one wishes to be able to compute the Euler characteristic from the chain groups alone, rather than the homology groups. Indeed, this would more closely mirror the definition of the Jones polynomial, in which one term is contributed by each vertex. It turns out that this becomes true when the differentials of the chain complex preserve the quantum degree.

Theorem 4.5. Let $\mathcal{C}$ be a chain complex with chain groups C_k . If the differentials of $\mathcal{C}$ preserve the quantum degree, we have that

$$\chi_q(\mathcal{C}) = \sum_m (-1)^m \mathrm{qdim} C_m$$

We are finally able to define chain complex that gives rise to Khovanov homology.

Let L be a link diagram with n_+ positive crossings and n_- negative crossings, and set $n = n_+ + n_-$. Let $c_1, \dots, c_n$ be the crossings. Finally, let V be the graded vector space with one basis element of quantum degree 1 and one basis element of quantum degree -1 , so $\mathrm{qdim} V = q + q^{-1}$.

Consider the 2^n -vertex cube Q of smoothings of L . We map this cube to another 2^n -vertex cube Q^* by sending a smoothing with height $h(r)$ and k loops to the vector space $V^{\otimes k}\{h(r)\}$. Then we construct the *Khovanov chain complex* $\mathcal{C}$ as follows. Let $\llbracket L \rrbracket_j$ be the direct sum of all vertices of Q^* with height j , and let $\llbracket L \rrbracket$ be the complex with chain groups $\llbracket L \rrbracket_j$ and differentials ∂_j that we will define momentarily. Then we set $\mathcal{C} = \llbracket L \rrbracket[-n_-]\{n_+ - 2n_-\}$.

This construction is quite natural – the quantum dimension of V is exactly the loop constant in our modified formulation of the Jones polynomial, and shifting the degree by $h(r)$ corresponds to the leading power of q in each term of the sum that yields the Jones polynomial. The final shift by $\{n_+ - 2n_-\}$ is to account for the corresponding normalization term in $\hat{J}(L)$. The other shift $[-n_-]$ is a bit harder to explain, but we do not concern ourselves with this detail in hopes of understanding the bigger picture.

The final step in the construction is to define the differentials of the complex. The obvious way to do this is to define a map $d_{r,s}$ for each edge rs of the cube, then take the differential

$\llbracket L \rrbracket_j \xrightarrow{\partial_{j+1}} \llbracket L \rrbracket_{j+1}$ to be the sum of all $d_{r,s}$ where r has height r and j has height $j+1$. As it turns out, this is the fundamental idea in the construction, but a small modification is needed. Recall that we must have $\partial_{j+1}\partial_j = 0$. Because each face of the cube contributes 4 terms to the sum $\partial_{j+1}\partial_j$, we could make the composition of maps along the edges of the face anti-commute to satisfy this property. However, making an explicit choice of maps to satisfy this anti-commutativity is a bit of a hassle, so we instead make the following clever observation.

Let r, s, t, u be the four vertices of a face of the cube, such that there exist maps $r \xrightarrow{d_1} s \xrightarrow{d_2} u$ and $r \xrightarrow{d_3} t \xrightarrow{d_4} u$. If we set $d_2 \circ d_1 = d_4 \circ d_3$, then when we take the summation over edge maps, we can multiply d_2 by a factor of -1 to ensure $\partial^2 = 0$. The problem then becomes how to decide which edges to attach a factor of -1 to. Since each edge e only changes one coordinate, we can write $e \in \{0, 1, \star\}^n$ where the $\star$ denotes the coordinate that is changed (there is exactly one $\star$ in every e). Then, if we let j be the coordinate of the $\star$ and define $(-1)^e = (-1)^{\sum_{i < j} e_i}$, we can simply take

$$\partial_{j+1} = \sum_{\substack{e=rs \\ h(s)=j+1}} (-1)^e d_e$$

Now we give the definitions of the maps themselves, noting that the only condition they must satisfy is that any face of the cube (viewed as a diagram) must commute.

To begin, we consider how one might interpret a map between one smoothing of a link L to an adjacent smoothing (that is, one that differs only at one crossing). One might guess that two adjacent smoothings are not too different, and this is true, even in a precise topological sense: to get from one smoothing to an adjacent one, it suffices to either merge two loops into one or split one loop into two. Recall that, in each vertex of our cube Q^* , each copy of V corresponds to a particular loop. Therefore, we define two linear maps $m : V \otimes V \rightarrow V$ (the merging map) and $s : V \rightarrow V \otimes V$ (the splitting map) as follows:

$$m : \begin{cases} v_+ \otimes v_+ \mapsto v_+ \\ v_+ \otimes v_- \mapsto v_- \\ v_- \otimes v_+ \mapsto v_- \\ v_- \otimes v_- \mapsto 0 \end{cases}$$

$$s : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

Then we set $d_e = m$ if e merges two loops, and $d_e = s$ if e splits a loop. Observe that these differentials preserve quantum degree.

Definition 4.6 (Khovanov homology). Let $\mathcal{C}$ be the Khovanov chain complex of a link diagram L defined above. The Khovanov homology of L is the homology of $\mathcal{C}$.

4.4 Properties of Khovanov Homology

Here, we present (without proof) some properties of Khovanov homology that make it so useful.

1. **The graded Euler characteristic of the Khovanov complex is the Jones polynomial.** This is easy to see from the definition, and was the motivating factor in the process of constructing the chain complex.

2. **Khovanov homology is strictly stronger than the Jones polynomial.** There exist non-isotopic knots that have the same Jones polynomial, but different Khovanov homologies. Interestingly, while it is not known whether the Jones polynomial detects the unknot, it is known that Khovanov homology does! (This unfortunately does not answer the question of whether it is possible to detect the unknot in *polynomial time* — even computing the Jones polynomial is known to be NP-hard, and since the Khovanov homology “contains” the Jones polynomial in some sense, computing it must also be NP-hard).

References

- [1] Dror Bar-Natan. On khovanov’s categorification of the jones polynomial. *Algebraic & Geometric Topology*, 2(1):337–370, May 2002.
- [2] Stephen J. Bigelow. Does the jones polynomial detect the unknot?, 2000.
- [3] Colleen Delaney, Eric C. Rowell, and Zhenghan Wang. Local unitary representations of the braid group and their applications to quantum computing, 2016.
- [4] Christian Kassel and Vladimir Turaev. *Braid Groups*. Springer New York, 2008.
- [5] Louis Kauffman. An introduction to khovanov homology, 2016.
- [6] P. B. Kronheimer and T. S. Mrowka. Khovanov homology is an unknot-detector, 2010.
- [7] Andrew Lobb. A feeling for khovanov homology. *Notices of the American Mathematical Society*, 71:1, 05 2024.
- [8] Jozef H. Przytycki. Khovanov homology: categorification of the kauffman bracket relation, 2005.
- [9] Melissa Zhang. Notes on khovanov homology, 2025.